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Algebraic Structures in the Decomposition of Mixed and Multiplicative Trend-Cycle Models

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***Abstract:** In his study examines the algebraic foundations of mixed and multiplicative models in the decomposition of trend-cycle components within time series analysis. By leveraging algebraic structures, we explore how these models interact with seasonal patterns and variance distribution. The Buys-Ballot table is utilized to assess changes in row, column, and overall means and variances, particularly in cases where no trend is present. Our findings provide a theoretical framework for distinguishing the structural properties of mixed and multiplicative models, enhancing their application in time series modelling and forecasting.*

***Keywords:** Mixed Models, Time Series Analysis, Buys-Ballot Table, Variance Distribution, Trend Analysis*

Introduction

Time series methods involve decomposing an observed time series into components that represent trend, seasonal, cyclical, and irregular (asymmetrical) patterns. When working with short periods of time series data, the cyclical component often becomes superimposed on the trend, blending with the observed time series $\left(\chi_r, r = 1, 2, ...,\right)$ can be decomposed into the trend-cycle component $\left(N_r\right)$, seasonal component $\left(\mu_r\right)$ and the irregular/residual component $\left(\lambda_r\right)$ [1]. As a result, the decomposition models are

Additive Model:

$$\chi_r = N_r + \mu_r + \lambda_r \tag{1}$$

Multiplicative Model:

$$\chi_r = N_r \times \mu_r \times \lambda_r \tag{2}$$

and Mixed Model

$$\chi_r = N_r \times \mu_r + \lambda_r . \tag{3}$$

While a seasonal impact exists, it is always considered to have an interval of s

$$\mu_{r+\tau} = \mu_r , \forall r \tag{4}$$

Additive model is assumed that, Over a full period/year

$$\sum_{r=0}^{\tau} \mu_{r+n} = 0 . \tag{5}$$

Similarly, in both multiplicative and mixed models,

$$\sum_{k=1}^{\tau} \mu_{r+k} = \tau . \tag{6}$$

Trend, seasonality, cycles, and residuals are the four separate components that make up the additive model, which is simpler to work with mathematically. It is assumed that during the time period under analysis, these factors have equivalent influence in absolute terms. This presumption usually remains true when the time series covers a brief period, the trend's growth or decrease rate is small, and no transformation is required. Nevertheless, for a more thorough examination, mixed and multiplicative models will also be taken into account in this study.

The comparison of time series decomposition using multiplicative and additive models [2]. In the additive model, they believe that the seasonal fluctuations have a constant amplitude in relation to the trend. A Chi-Square test based on the seasonal variances of the Buys-Ballot table was devised since, in contrast, the multiplicative model's seasonal fluctuations' amplitude is dependent on the trend [3–4]. In time series analysis, this test has been shown to be extremely effective and successful in differentiating between multiplicative and mixed models.

The expected mean values of mixed and multiplicative

The row mean for mixed model is;

$$\bar{\chi}_i = \alpha - \beta\tau(k-1) + \frac{\beta}{\tau}\sum_{k=1}^{\tau} k\mu_k + \bar{\lambda}_r \tag{7}$$

$$E\left(\bar{X}_i\right) = E\left[a - bs(i-1) + \frac{b}{s}\sum_{j=1}^{s} jS_j\right] + E\left(\bar{e}_i\right)$$

Hence, the expected value of row mean is

$$E\left(\bar{X}_i\right) = a - bs + bsi + \frac{b}{s}\sum_{j=1}^{s} jS_j \tag{8}$$

For multiplicative model, the row mean is

$$\bar{X}_i = \left[a - bs(i-1) + \frac{b}{s}\sum_{j=1}^{s} jS_j + bsi\right]\bar{e}_i \tag{9}$$

Thus, the expected value of row mean is

$$E\left(\bar{X}_i\right) = a - bs + bsi + \frac{b}{2}\sum_{j=1}^{s} jS_j \tag{10}$$

The column mean for mixed model is

$$\bar{X}_{.j} = \left[a + b\left(\frac{n-s}{2}\right) + bj\right]S_j + \bar{e}_{.j}$$

$$E\left(\bar{X}_{.j}\right) = E\left[a + b\left(\frac{n-s}{2}\right) + bj\right]E\left(S_j\right) + E\left(\bar{e}_{.j}\right) \tag{11}$$

Therefore, the expected value of the column mean is

$$E\left(\bar{X}_{.j}\right) = \left[a + b\left(\frac{n-s}{2}\right) + bj\right]S_j \tag{12}$$

for multiplicative model, the column mean is

$$\bar{X}_{.j} = \left[a\bar{e}_{.j} + \frac{bs}{m}\sum_{i=1}^{m} ie_{ij} - bs\bar{e}_{.j} + bj\bar{e}_{.j}\right]S_j \tag{13}$$

$$E\left(\bar{X}_{.j}\right) = E\left[a\bar{e}_{.j} + \frac{bs}{m}\sum_{i=1}^{m}(i-1) - bs + bj\right]E\left(S_j\right)$$

$$= \left[a + \frac{bsm}{m}\frac{(m-1)}{2} - bs + bj\right]S_j$$

Hence, the expected value of column mean is

$$E\left(\bar{X}_{.j}\right) = \left[a + b\left(\frac{n-s}{2}\right) + bj\right]S_j \tag{14}$$

Overall mean for mixed model is

$$\bar{X}_{..} = a + b\left(\frac{n-s}{2}\right) + bc_1 + \bar{e}_{..}$$

$$E\left(\bar{X}_{..}\right) = E\left[a + b\left(\frac{n-s}{2}\right) + bc_1\right] + E\left(\bar{e}_{..}\right) \tag{15}$$

Hence , the expected value for the overall mean is

$$E\left(\bar{X}_{..}\right) = a + b\left(\frac{n-s}{2}\right) + bc_1 \tag{16}$$

Estimation of Trend Parameters and Seasonal Indices

We use the grand and periodic means to establish the characteristics of the trend line. We suppose that the length of the periodic interval is s. Both multiplicative and mixed models are obtained by applying the expressions in (7) and (9).

$$\bar{X}_{i.} = a - b(s - c_1) + (bs)i \qquad (17)$$

$$\bar{X}_{i.} = a + \bar{e}_i \qquad (21)$$

For multiplicative model, when b=0, that is when there is no trend,

$$\bar{X}_{i.} = a \qquad (22)$$

Estimation of $S_j, \quad j = 1,2,\ldots,s$

To estimate the seasonal indices, the seasonal and grand means are employed. It is also assumed that the length of the periodic interval is s. The expressions in (11) and (13) yield both multiplicative and mixed models.

$$\bar{X}_j = \left[a + b\left(\frac{n-s}{2}\right) + bj\right]S_j \qquad (23)$$

Where. $\alpha = a + b\left(\frac{n-s}{2}\right)$ (24)

For mixed model, where there is no trend (b=0), we obtain from (11)

$$\hat{S}_j = \frac{\bar{X}_j}{a + \bar{e}_j} \qquad (25)$$

For multiplicative model, when b=0, that is when there is no trend, we obtain from (13)

$$\hat{S}_j = \frac{\bar{X}_{.j}}{a\bar{e}_{.j}},$$ Linear trend-cycle component: $M_t = a + b_t, \quad t = 1,2,\ldots,n = ms$

Table 1: Expected values of means for multiplicative and mixed models

Measures	Multiplicative model	Mixed model
$\bar{X}_{i.}$	$[a - bs + bsi] + \frac{b}{s}\sum_{j=1}^{s} jS_j$	$a - bs + bsi + \frac{b}{s}\sum_{j=1}^{s} jS_j$
$\bar{X}_{.j}$	$\left[a + b\left(\frac{n-s}{2}\right) + bj\right]S_j$	$\left[a + b\left(\frac{n-s}{2}\right) + bj\right]S_j$
$\bar{X}_{..}$	$a + b\left(\frac{n-s}{2}\right) + bc_1$	$a + b\left(\frac{n-s}{2}\right) + bc_1$

Where, $c_1 = \frac{1}{s}\sum_{j=1}^{s} jS_j$

Table 2: Estimation of Trend Parameters and Seasonal Indices

Parameter	Multiplicative model	Mixed model
a	$a + \hat{b}(s - c_1)$	$a + \hat{b}(s - c_1)$
b	$\frac{\beta}{s}$	$\frac{\beta}{s}$
S_j	$\frac{\bar{X}_{.j}}{a + b\left(\frac{n-s}{2}\right) + b_j}$	$\frac{\bar{X}_{.j}}{a + b\left(\frac{n-s}{2}\right) + b_j}$

Table 3: Trend and seasonal indices estimate

Parameter	Multiplicative model	Mixed model
$\bar{X}_{i.}$	α	$a + \bar{e}_{.i}$
$\bar{X}_{.j}$	β	$a + \bar{e}_{.i}$
$\bar{X}_{..}$	γ	$a + \bar{e}_{..}$
S_j	$\frac{\bar{X}_{.j}}{a\bar{e}_{.j}}$	$\frac{\bar{X}_{.j}}{a + \bar{e}_{.j}}$

Numerical Experiment

The goal of this section is to present a real-world example using monthly time series data on baptisms collected from Assumpta Cathedral in Owerri, Imo State, Nigeria, from 2009 to 2018, as shown in Appendix A. Figures 1 and 2 illustrate the time series plots of both the original and modified datasets. For both mixed and multiplicative models, the linear trend and seasonal indices are represented as follows:

$$\bar{X}_{.j} = 2.678 + 0.0221j \tag{29}$$

Using (23),(24) and (25)

$$\hat{b} = 0.0221, \qquad \hat{a} = 2.678 - 0.0221\left(\frac{120-12}{2}\right)$$

Table 4: Estimates of trend parameters

Parameter	Mixed model values	Multiplicative model values
a	1.5184	1.5184
b	0.0221	0.0221

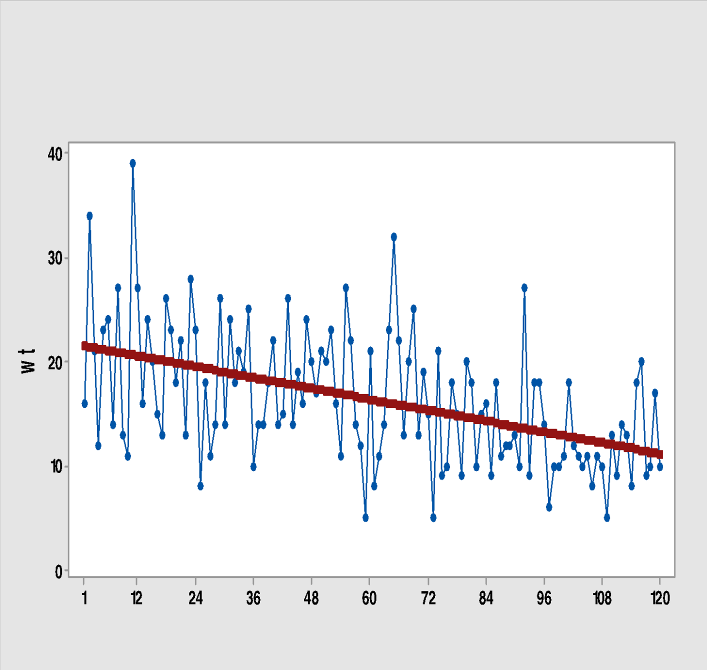

Figure 1: The actual series' time plot

$\hat{a} = 1.5184$, $\hat{S}_j = \dfrac{\bar{X}._j}{2.687 + 0.0221_j}$

Table 5: Analysis of Seasonal Indices

j	$\bar{X}_j$	$\hat{S}_j$
1	2.2481	0.8678
2	2.8372	1.0735
3	2.4543	0.9855
4	2.7235	1.2140
5	2.8670	1.0680
6	2.7110	1.0024
7	2.6860	1.0225
8	2.8392	1.1702
9	2.7190	1.0321
10	2.536	1.0021
11	2.8780	1.0263
12	2.7500	1.9734
$\sum_{j=1}^{s} \hat{S}_j$	32.2493	13.4378

Table 6: Analysis of Trend and Seasonal Indicator Parameters

Parameter	Multiplicative model values	Mixed model values
$\hat{a}$	1.5003	1.5003
$\hat{b}$	0.0301	0.0301
$\hat{S}_1$	0.7594	0.7594
$\hat{S}_2$	1.0853	1.0853
$\hat{S}_3$	1.003	1.003
$\hat{S}_4$	0.992	0.992
$\hat{S}_5$	1.368	1.368
$\hat{S}_6$	1.0004	1.0004
$\hat{S}_7$	0.9987	0.9987
$\hat{S}_8$	0.9876	0.9876
$\hat{S}_9$	0.8756	0.8756
$\hat{S}_{10}$	1.003	1.003
$\hat{S}_{11}$	1.007	1.007
$\hat{S}_{12}$	1.005	1.005
$\sum_{j=1}^{s} \hat{S}_j$	13.6154	13.6154

Note: mixed satisfies

$\left(\sum_{j=1}^{s} S_j = s\right)$ as in (6) Also, multiplicative model satisfies $\left(\sum_{j=1}^{s} S_j = s\right)$ as in (6)

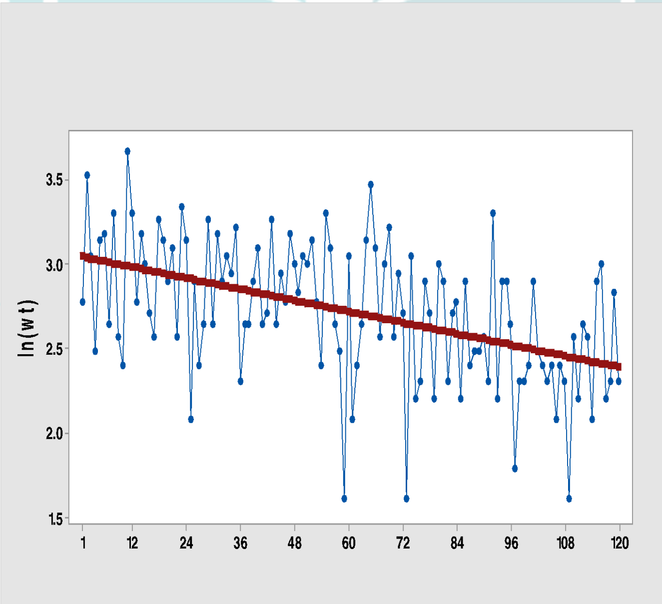

Figure 2: Time plot of the transformed series

Summary and Conclusion

Outcomes of the study show that the means and variances of the Buys-Ballot table for mixed and multiplicative models are different, the ordinary values of the means are the same for mixed as well as multiplicative models , the calculated values of estimated trend parameters and seasonal indices are the same for the two models, but different when there is no trend.

The column variances ($\hat{\sigma}_j^2$) of the Buys-Ballot table depends on the season j only through the square of the seasonal effect S_j^2 for mixed model. A quadratic function of the column j and square of the seasonal effect S_j^2 for multiplicative model.

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